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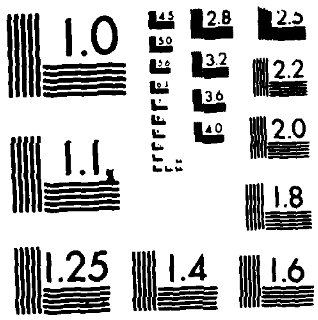

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SIMPLE BOUNDS FOR SOLUTIONS OF MONOTONE COMPLEMENTARITY PROBLEMS AND CONVEX PROGRAMS

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ABSTRACT

For a solvable monotone complementarity problem ~~we show~~ that each feasible point which is not a solution of the problem provides simple numerical bounds for some or all components of all solution vectors. Consequently for a solvable differentiable convex program each primal-dual feasible point which is not optimal provides simple numerical bounds for some or all components of all primal-dual solution vectors. ~~We~~ also give an existence result and simple bounds for solutions of monotone complementarity problems satisfying a new, distributed constraint qualification. This result carries over to a simple existence and boundedness result for differentiable convex programs satisfying a similar constraint qualification.

AMS (MOS) Classification: 90C30, 90C25

Key Words: Nonlinear programming, complementarity, monotonicity, convexity, bounds

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SIGNIFICANCE AND EXPLANATION

Simple bounds are given for solutions of fundamental optimization problems: monotone complementarity problems and convex programs. It is shown that each nonoptimal but feasible point carries within it simple numerical information which bounds some or all components of all solution vectors. Thus bounds are obtained without solving the problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SIMPLE BOUNDS FOR SOLUTIONS OF MONOTONE COMPLEMENTARITY PROBLEMS AND CONVEX PROGRAMS

O. L. Mangasarian, L. McLinden

1. The Monotone Complementarity Problem

This work is based on an extremely simple, but apparently unnoticed, property of the monotone complementarity problem [2,5,8,11,12] of finding a (z,w) in the 2k-dimensional Euclidean space R^{2k} such that

$$w = F(z) \geq 0, \; z \geq 0, \; zw = 0 \tag{1.1}$$

Here $F\colon D \to R^k$ is a monotone function on D where $R^k_+ \subset D \subset R^k$, that is

$$(z^2 - z^1)(F(z^2) - F(z^1)) \geq 0 \quad \text{for all} \quad z^1, z^2 \in D$$

The property is the following:

1.1 Theorem Let (z,w) be some feasible point of a solvable monotone complementarity problem (1.1), that is $w = F(z) \geq 0$, $z \geq 0$. Any solution $(\bar{z},\bar{w})$ of (1.1) is bounded as follows:

(a) $$\|\bar{z}_I\|_1 := \sum_{i\in I} \bar{z}_i \leq zw / \min w_{i\in I} := zw / \min_{i\in I} w_i$$

(b) $$\|\bar{w}_J\|_1 \leq zw / \min z_{i\in J}$$

(c) $$\|\bar{z}_I, \bar{w}_J\|_1 \leq zw / \min \{z_{i\in J}, w_{i\in I}\}$$

where $I = \{i \mid w_i > 0\}$ and $J = \{i \mid z_i > 0\}$.

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<u>Proof</u> For any solution $(\bar{z},\bar{w})$ of (1.1) we have by the monotonicity of F and $\bar{z}\bar{w} = 0$ that

$$zw \geq \bar{z}w + z\bar{w}$$

Hence by the nonnegativity of (z,w) and $(\bar{z},\bar{w})$ we have

(a) $\qquad zw \geqq \bar{z}_I w_I \geqq \|\bar{z}_I\|_1 \min w_{i\in I}$

(b) $\qquad zw \geqq z_J \bar{w}_J \geqq \|\bar{w}_J\|_1 \min z_{i\in J}$

(c) $\qquad zw \geqq \bar{z}_I w_I + z_J \bar{w}_J \geqq \|\bar{z}_I, \bar{w}_J\|_1 \cdot \min \{z_{i\in J}, w_{i\in I}\}$ $\qquad \square$

Theorem 1.1 is a partial extension to the monotone complementarity problem of a corresponding result, Theorem 2.2 of [7], for the positive semidefinite linear complementarity problem. Note that, unlike the linear case, feasibility for the nonlinear monotone complementarity problem does not imply solvability as shown by the simple example of [10].

Theorem 1.1 shows that <u>any</u> feasible point (z,w) of a solvable monotone complementarity problem (1.1) which is not a solution of the problem (so that at least I is nonempty or J is nonempty) provides <u>some</u> information about the magnitude of the solution set. In certain cases, such as when $w > 0$, we get a bound on all components of all solution vectors $\bar{z}$.

With the bounds given by Theorem 1.1 it is possible to obtain bounds for optimal solutions and multipliers of solvable differentiable convex programs once they are cast as monotone complementarity problems. (See Section 2.) But before doing that we show how the bounds of Theorem 1.1 can be extended to approximate solutions of monotone complementarity problems which may not even be solvable. Let

$$(1.2) \qquad \alpha := \inf\{zw \mid w = F(z) \geq 0,\ z \geq 0\} \geq 0,$$

and for $\varepsilon \geq 0$ let $(\bar{z}(\varepsilon), \bar{w}(\varepsilon))$ be an ε-solution of the optimization problem of (1.2), that is

$$(1.3) \qquad \bar{w}(\varepsilon) = F(\bar{z}(\varepsilon)) \geq 0,\ \bar{z}(\varepsilon) \geq 0,\ \alpha + \varepsilon \geq \bar{z}(\varepsilon)\bar{w}(\varepsilon) \geq \alpha$$

Note that for any $\varepsilon > 0$, an ε-solution always exists provided problem (1.2) has at least one feasible point. A 0-solution exists provided the infimum of (1.2) is attained, that is the infimum is a minimum. Furthermore if $\alpha = 0$, then an ε-solution of (1.2) is an "approximate" solution of the complementarity problem (1.1) which is an exact solution if $\varepsilon = 0$. With these concepts in mind it follows from (1.2), (1.3) and the monotonicity of F that, for any feasible (z,w) and $\varepsilon \geq 0$,

$$(1.4) \qquad 2zw + \varepsilon \geq zw + \alpha + \varepsilon \geq zw + \bar{z}(\varepsilon)\bar{w}(\varepsilon) \geq z\bar{w}(\varepsilon) + \bar{z}(\varepsilon)w$$

Consequently, arguing exactly as in Theorem 1.1 we obtain the following bounds for ε-solutions of the optimization problem (1.2).

1.2 <u>Theorem</u> Let F be monotone on R^k_+ and let (z,w) be some feasible point of the optimization problem of (1.2), that is, $w = F(z) \geq 0$, $z \geq 0$, and let $\varepsilon \geq 0$. Any ε-solution $(\bar{z}(\varepsilon), \bar{w}(\varepsilon))$ of (1.2), defined by (1.3), is bounded as follows:

(a) $\qquad \|\bar{z}_I(\varepsilon)\|_1 \leq (zw + \alpha + \varepsilon)/\min w_{i\epsilon I} \leq (2zw + \varepsilon)/\min w_{i\epsilon I}$

(b) $\qquad \|\bar{w}_J(\varepsilon)\|_1 \leq (zw + \alpha + \varepsilon)/\min z_{i\epsilon J} \leq (2zw + \varepsilon)/\min z_{i\epsilon J}$

(c) $\quad \|\bar{z}_I(\varepsilon), \bar{w}_J(\varepsilon)\|_1 \leq (zw + \alpha + \varepsilon)/\min\{z_{i\epsilon J}, w_{i\epsilon I}\} \leq (2zw + \varepsilon)/\min\{z_{i\epsilon J}, w_{i\epsilon I}\}$

where $I = \{i \mid w_i > 0\}$ and $J = \{i \mid z_i > 0\}$.

Note that this theorem subsumes Theorem 1.1. For if we assume that the complementarity problem (1.1) is solvable as in Theorem 1.1, then we can set $\alpha = \varepsilon = 0$ in Theorem 1.2 and obtain Theorem 1.1.

A generalization of Theorem 1.2 is possible if, instead of one feasible point (z,w), we consider p feasible points (z^j, w^j), $j=1,2,\ldots,p$, of the optimization problem of (1.2) and correspomding weights $\lambda^j \geq 0$, $j=1,\ldots,p$, such that $\sum_{j=1}^{p} \lambda^j = 1$. Then by (1.2), (1.3) and the monotonicity of F we have that

$$(1.5)\quad 2\sum_{j=1}^{p} \lambda^j z^j w^j + \varepsilon \geq \sum_{j=1}^{p} \lambda^j z^j w^j + \alpha + \varepsilon \geq \sum_{j=1}^{p} \lambda^j z^j w^j + \bar{z}(\varepsilon)\bar{w}(\varepsilon) \geq \sum_{j=1}^{p} \lambda^j z^j \bar{w}(\varepsilon) + \sum_{j=1}^{p} \lambda^j \bar{z}(\varepsilon) w^j$$

Again arguing as in Theorem 1.1 we obtain the following bounds.

1.3 <u>Theorem</u> Let F be monotone on R^k_+ and let (z^j, w^j), $j=1,2,\ldots,p$, be feasible points of the optimization problem of (1.2), that is, $w^j = F(z^j) \geq 0$, $z^j \geq 0$, $j=1,2,\ldots,p$. Let $\lambda^j \geq 0$, $j=1,2,\ldots,p$, $\sum_{j=1}^{p} \lambda^j = 1$ and let $\varepsilon \geq 0$. Any ε-solution $(\bar{z}(\varepsilon), \bar{w}(\varepsilon))$ of (1.2) defined by (1.3) is bounded as follows:

(a) $$\|\bar{z}_I(\varepsilon)\|_1 \leq \Big(\sum_{j=1}^{p} \lambda^j z^j w^j + \alpha + \varepsilon\Big)/\min \hat{w}_{i\in I} \leq \Big(2\sum_{j=1}^{p} \lambda^j z^j w^j + \varepsilon\Big)/\min \hat{w}_{i\in I}$$

(b) $$\|\bar{w}_J(\varepsilon)\|_1 \leq \Big(\sum_{j=1}^{p} \lambda^j z^j w^j + \alpha + \varepsilon\Big)/\min \hat{z}_{i\in J} \leq \Big(2\sum_{j=1}^{p} \lambda^j z^j w^j + \varepsilon\Big)/\min \hat{z}_{i\in J}$$

(c) $$\|\bar{z}_I(\varepsilon), \bar{w}_J(\varepsilon)\|_1 \leq \Big(\sum_{j=1}^{p} \lambda^j z^j w^j + \alpha + \varepsilon\Big)/\min\{\hat{z}_{i\in J}, \hat{w}_{i\in I}\} \leq \Big(2\sum_{j=1}^{p} \lambda^j z^j w^j + \varepsilon\Big)/\min\{z_{i\in J}, w_{i\in I}\}$$

where $I = \{i \mid \hat{w}_i > 0\}$, $J = \{i \mid \hat{z}_i > 0\}$, $\hat{z} := \sum_{j=1}^{p} \lambda^j z^j$ and $\hat{w} := \sum_{j=1}^{p} \lambda^j w^j$.

We note that the first inequality of each of (a), (b) and (c) of Theorem 1.3 remains valid even if we do not require that $w^j \geq 0$ and $z^j \geq 0$, but merely that $z^j \in D$, where $R^k_+ \subset D \subset R^k$, F is monotone on D and $\hat{z} \geq 0$ and $\hat{w} \geq 0$. This remark will be employed in Theorem 1.4.

An application of Theorem 1.3 is the following boundedness result for complementarity problems satisfying a new, "distributed" constraint qualification, and for which we also establish an existence result.

1.4 Theorem (Existence and boundedness of solutions of monotone complementarity problems under a distributed constraint qualification) Let $F\colon D \to R^k$ be monotone and continuous on D such that $R^k_+ \subset D \subset R^k$, let $z^j \in D$, $w^j = F(z^j) \in R^k$, $j=1,2,\ldots,p$, be such that $\hat{z} := \sum_{j=1}^{p} \lambda^j z^j \geq 0$, $\hat{w} := \sum_{j=1}^{p} \lambda^j w^j > 0$ for some $\lambda^j \geq 0$, $j=1,2,\ldots p$, $\sum_{j=1}^{p} \lambda_j = 1$. Then the complementarity problem (1.1) is solvable. Any solution $(\bar{z},\bar{w})$ is bounded as follows:

$$\|\bar{z}\|_1 \leq \left(\sum_{j=1}^{p} \lambda^j z^j w^j\right) \Big/ \min_{1\leq i\leq k} \hat{w}_i \tag{1.6}$$

Proof The bound (1.6) follows from Theorem 1.3(a) with $\alpha = \varepsilon = 0$ and the remark following it, once we have established the existence of a solution to the complementarity problem (1.1), which we proceed to do now by means of the Brouwer fixed point theorem [1,14]. Let

$$C := \{z \mid z \geq 0,\ \hat{w}z \leq \hat{w}\hat{z} + \gamma\},$$

where

$$\gamma > \max\,\{1, -\hat{w}\hat{z} + \sum_{j=1}^{p} \lambda^j z^j w^j\} \geq 1 \tag{1.7}$$

The set C is nonempty, compact and convex and the single-valued mapping [4] defined by the 2-norm projection of $z - F(z)$ on C:

$$z \to \operatorname*{argmin}_{y\in C} \|y - z + F(z)\|_2$$

defines a continuous function from C into itself. Hence by Brouwer's

theorem this function must have a fixed point $\bar{z} \in C$. Such a point satisfies the minimum principle optimality condition [6].

(1.8) $$\bar{z} \in C,\ F(\bar{z})(y - \bar{z}) \geq 0 \quad \forall y \in C$$

If $\hat{w}\bar{z} < \hat{w}\hat{z} + \gamma$ then $\bar{z}$ solves (1.1). Indeed $\bar{z} + \delta e_i$, $i=1,2,\ldots,k$, is in C for δ sufficiently small and positive and e_i the ith unit coordinate vector, and hence by (1.8) it follows that $F(\bar{z}) \geq 0$, $\bar{z} \geq 0$, and $\bar{z}F(\bar{z}) \leq 0$ by taking $y = 0$ in (1.8). We now show that the case

(1.9) $$\hat{w}\bar{z} = \hat{w}\hat{z} + \gamma$$

cannot occur. For if it did, then from the monotonicity of F we have

$$\bar{z}F(\bar{z}) \geq -z^j w^j + z^j \bar{w} + \bar{z}w^j, \quad j=1,2,\ldots,p$$

where $\bar{w} := F(\bar{z})$. Multiplying by λ^j and summing over j gives

$$\bar{z}F(\bar{z}) \geq \sum_{j=1}^{p} -\lambda^j z^j w^j + \hat{z}\bar{w} + \bar{z}\hat{w}$$

$$> \hat{z}\bar{w} = \hat{z}F(\bar{z}) \quad \text{(By (1.9) and (1.7))}$$

Hence $F(\bar{z})(\hat{z} - \bar{z}) < 0$ which contradicts (1.8). So (1.9) cannot occur and $\bar{z}$ solves (1.1). $\square$

We note that the existence part of the above theorem for the ordinary constraint qualification, that is $p = 1$, was obtained by Moré [12, Theorem 3.2] and by one of the authors in [8, Theorem 1] for the case of multivalued monotone mappings.

It is interesting to note that the complementarity problem of Megiddo [10] which has no solution does not satisfy the distributed constraint

qualification of Theorem 1.4 and hence demonstrates the sharpness of that condition. On the other hand Theorem 1.3(a) can be used to give an exact upper bound on the bounded component of the solution of problem (1.2) for Megiddo's example.

We also note the distributed constraint qualification of Theorem 1.4 is implied by the ordinary constraint qualification if we take $p = 1$. The converse is true if $D = R^k_+$ and F is concave on R^k_+. However F is not concave in general, and in fact is merely monotone when it is derived from a differentiable convex program. (See Section 2.) However for the general case of a monotone F and $D = R^k_+$, it can be shown [9] that the two constraint qualifications are equivalent. Nevertheless the distributed qualification may be easier to verify.

2. Bounds for Solutions of Convex Programs

We consider now the solvable differentiable convex program

$$(2.1) \qquad \min_x f(x) \quad \text{s.t. } y = -g(x) \geq 0,\ x \geq 0$$

where $f: R^n \to R$, $g: R^n \to R^m$ are convex and differentiable functions, together with its dual [6]

$$(2.2) \qquad \max_{x,u} L(x,u) - x\nabla_x L(x,u) \quad \text{s.t. } v = \nabla_x L(x,u) \geq 0,\ u \geq 0$$

where $L(x,u)$ is the standard Lagrangian

$$L(x,u) = f(x) + ug(x)$$

and ∇_x denotes the gradient vector with respect to x. We note that the Karush-Kuhn-Tucker conditions

$$(2.3) \qquad \begin{aligned} v &= \nabla_x L(x,u) = \nabla f(x) + u\nabla g(x) \geq 0,\ x \geq 0,\ xv = 0 \\ y &= -\nabla_u L(x,u) = -g(x) \geq 0,\ u \geq 0,\ uy = 0 \end{aligned}$$

hold if and only if (x,y,u,v) solves the dual programs (2.1)-(2.2) with equal extrema [6]. If the constraints of (2.1) satisfy the Slater constraint qualification, that is $g(x) < 0$ for some $x \geq 0$, then for each solution of (2.1) the Karush-Kuhn-Tucker conditions (2.3) are satisfiable [6]. If we make the definitions

$$(2.4) \qquad z := \begin{pmatrix} x \\ u \end{pmatrix},\quad w := \begin{pmatrix} v \\ y \end{pmatrix},\quad F(z) := \begin{pmatrix} \nabla_x L(x,u) \\ -\nabla_u L(x,u) \end{pmatrix}$$

then the Karush-Kuhn-Tucker conditions take on the equivalent complementarity problem formulation [2]

$$w = F(z) \geq 0, \; z \geq 0, \; zw = 0 \tag{2.5}$$

Note that the "twisted" derivative involved in the definition of $F(z)$ has also been used in [3,13,5,8]. We now establish the monotonicity of this $F(z)$.

2.1 <u>Lemma</u> Let f and g be differentiable and convex on R^n and let $F(z)$ be defined as in (2.4). Then $F(z)$ is monotone and continuous for all $z \in R^n \times R^m_+$.

<u>Proof</u> By the convexity of g and $\bar{u} \geq 0$, $u \geq 0$ we have that

$$\bar{u}\,(g(x) - g(\bar{x})) \geq \bar{u}\nabla g(\bar{x})(x - \bar{x})$$

$$-u(-g(\bar{x}) + g(x)) \geq -u\nabla g(x)(-\bar{x} + x)$$

Addition of these two inequalities gives

$$-\,(u - \bar{u})(g(x) - g(\bar{x})) \geq (\bar{u}\nabla g(\bar{x}) - u\nabla g(x))(x - \bar{x}) \tag{2.6}$$

Hence

$$\begin{aligned}(z - \bar{z})(F(z) - F(\bar{z})) &= \begin{pmatrix} x - \bar{x} & u - \bar{u} \end{pmatrix} \begin{pmatrix} \nabla_x L(x,u) - \nabla_x L(\bar{x},\bar{u}) \\ -\,(g(x) - g(\bar{x})) \end{pmatrix} \\ &\geq (x-\bar{x})(\nabla f(x) - \nabla f(\bar{x})) \qquad \text{(By (2.6))} \\ &\geq 0 \qquad \text{(By convexity of } f\text{)}\end{aligned}$$

The continuity of F follows from the fact that a differentiable convex function on R^n is continuously differentiable. □

We can now apply Theorem 1.1 to the monotone function $F(z)$ of (2.4) to obtain bounds for optimal solutions and multipliers of (2.1).

2.2 <u>Theorem</u> Let f and g be differentiable and convex on R^n. Each primal-dual feasible point of (2.1)-(2.2), that is (x,y,u,v) satisfying

$$y = -g(x) \geq 0,\ x \geq 0,\ v = \nabla_x L(x,u) \geq 0,\ u \geq 0,$$

bounds any point $(\bar{x},\bar{y},\bar{u},\bar{v})$ which solves the primal-dual programs (2.1)-(2.2) with equal extrema, or equivalently, which satisfies the Karush-Kuhn-Tucker conditions (2.3) for (2.1) as follows:

(a) $$\sum_{i\in I_1} \bar{x}_i =: \|\bar{x}_{I_1}\|_1 \leq (xv+uy)/\min v_{i\in I_1}$$

(b) $$\|\bar{y}_{J_2}\|_1 \leq (xv+uy)/\min u_{i\in J_2}$$

(c) $$\|\bar{u}_{I_2}\|_1 \leq (xv+uy)/\min y_{i\in I_2}$$

(d) $$\|\bar{v}_{J_1}\|_1 \leq (xv+uy)/\min x_{i\in J_1}$$

where

$$I_1 = \{i|v_i>0\},\ J_2 = \{i|u_i>0\},\ I_2 = \{i|y_i>0\},\ J_1 = \{i|x_i>0\}$$

<u>Proof</u> Immediate from Theorem 1.1, Lemma 2.1 and definition (2.4). □

Theorem 2.2 is a partial extension of Theorem 3.1 of [7] where bounds for solutions of linear programs were given.

All the other theorems of Sectiom 1 apply in a straightforward manner to the convex program (2.1) via the complementarity formulation (2.4)-(2.5). We state below the counterpart of Theorem 1.4 for the convex program (2.1).

2.3 <u>Theorem</u> (Existence and boundedness of solutions of differentiable convex programs under a distributed constraint qualification) Let f and g be differentiable and convex on R^n, let

$$y^j = -g(x^j) \in R^m,\ x^j \in R^n,\ v^j = \nabla_x L(x^j,u^j) \in R^n,\ u^j \geq 0,\ j=1,2,\ldots,p$$

be such that for some $\lambda^j \geq 0$, $j=1,2,\ldots,p$, $\sum_{j=1}^{p} \lambda_j = 1$:

$$\hat{x} := \sum_{j=1}^{p} \lambda^j x^j \geq 0,\ \hat{y} := \sum_{j=1}^{p} \lambda^j y^j > 0,\ \hat{v} := \sum_{j=1}^{p} \lambda^j v^j > 0$$

Then there exists $(\bar{x},\bar{y},\bar{u},\bar{v})$ which solves the dual programs (2.1)-(2.2) with equal extrema. Any such solution $(\bar{x},\bar{y},\bar{u},\bar{v})$ is bounded as follows:

$$\|\bar{x},\bar{u}\|_1 \leq \left(\sum_{j=1}^{p} \lambda^j (x^j v^j + u^j y^j)\right) \Big/ \min_{\substack{1\leq i\leq n \\ 1\leq \ell\leq m}} \{\hat{v}_i, \hat{y}_\ell\}$$

Note that the requirement $u^j \geq 0$ in Theorem 2.3 is made because the monotonicity of F of Lemma 2.1 is established only on $R^n \times R^m_+$ and not on $R^n \times R^m$.

We give now a simple example illustrating the bounds of Theorem 2.2.

2.4 <u>Example</u> $\min\ x_1 + x_2 \quad \text{s.t. } y = x_2 - e^{x_1} \geq 0,\ x_1, x_2 \geq 0$

The dual problem is

$$\begin{array}{ll} \max & x_1 + x_2 - u(x_2 - e^{x_1}) - vx \\ \text{s.t.} & v_1 = 1 + ue^{x_1} \geq 0 \\ & v_2 = 1 - u \quad\ \geq 0 \\ & \qquad\quad u \quad\ \geq 0 \end{array}$$

The primal-dual solution is $\bar{x}_1 = 0,\ \bar{x}_2 = 1,\ \bar{y} = 0,\ \bar{u} = 1,\ \bar{v}_1 = 2,\ \bar{v}_2 = 0$.

(a) To get a bound on $\|\bar{x}\|_1$, take $x_1 = \alpha \geq 0$, $x_2 = e^\alpha$ and $u = 0$. Hence $y = 0$, $v_1 = 1$, $v_2 = 1$, $xv + uy = \alpha + e^\alpha$ and

$$1 = \|\bar{x}\|_1 \leq \inf_{\alpha\geq 0} \alpha + e^\alpha = 1$$

(b) To get a bound on $\|\bar{y}\|_1$, take $x_1 = \alpha \geq 0$, $x_2 = e^\alpha$ and $u = 1$. Hence $y = 0$, $v_1 = 1 + e^\alpha$, $v_2 = 0$, $xv + uy = \alpha(1 + e^\alpha)$ and

$$0 = \|\bar{y}\|_1 \leq \inf_{\alpha \geq 0} \alpha(1 + e^{\alpha}) = 0$$

(c) To get a bound on $\|\bar{u}\|_1$, take $x_1 = 1 > 0$, $x_2 = \alpha + e$, $\alpha > 0$ and $u = 1$. Hence $y = \alpha$, $v_1 = 1 + e$, $v_2 = 0$, $xv + uy = 1 + e + \alpha$ and

$$1 = \|\bar{u}\|_1 \leq \inf_{\alpha > 0} \frac{1+e+\alpha}{\alpha} = 1$$

(d) To get a bound on $\|\bar{v}\|_1$, take $x_1 = \alpha > 0$, $x_2 = e^{\alpha}$ and $u = 1$. Hence $y = 0$, $v_1 = 1 + e^{\alpha}$, $v_2 = 0$, $xv + uy = \alpha(1 + e^{\alpha})$, and

$$2 = \|\bar{v}\|_1 \leq \inf_{\alpha > 0} \frac{\alpha(1+e^{\alpha})}{\alpha} = 2$$

We conclude by remarking that extensions of the results in this paper can also be established for the more general case in which the continuous monotone function F is replaced by a maximal monotone multifunction. Such extensions allow us to handle problem (2.1) with f and g nondifferentiable, convex and possibly taking the value of $+\infty$. Further extensions can also be proved in which R^k is replaced, for example, by any reflexive Banach space and R^k_+ is replaced by a closed convex cone satisfying certain interiority/linearity properties.

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For a solvable monotone complementarity problem we show that each feasible point which is not a solution of the problem provides simple numerical bounds for some or all components of all solution vectors. Consequently for a solvable differentiable convex program each primal-dual feasible point which is not optimal provides simple numerical bounds for some or all components of all primal-dual solution vectors. We also give an existence result and simple bounds for solutions of monotone complementarity problems satisfying a new, distributed constraint qualification. This result carries over to a simple existence and boundedness result for differentiable convex programs satisfying a similar constraint qualification.

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